

# POINCARÉ TYPE INEQUALITIES FOR TWO DIFFERENT BILATERAL GRAND LEBESGUE SPACES

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Abstract.

In this paper we obtain the non-asymptotic inequalities of Poincaré type between function and its weak gradient belonging the so-called Bilateral Grand Lebesgue Spaces over general metric measurable space. We also prove the sharpness of these inequalities.

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## 1. INTRODUCTION

Let  $(X, M, \mu, d)$  be metric measurable space with finite non-trivial measure  $\mu : 0 < \mu(X) < \infty$  and also with finite non-trivial distance function  $d = d(x, y) :$

$$0 < \text{diam}(X) := \sup_{x, y \in X} d(x, y) < \infty.$$

Define also for arbitrary numerical measurable function  $u : X \rightarrow R$  the following average

$$u_X = \frac{1}{\mu(X)} \int_X u(x) d\mu(x),$$

$$\|u\|_p = \left[ \int_X |u(x)|^p d\mu(x) \right]^{1/p}, \quad p = \text{const} \in [1, \infty],$$

$g(x) = \nabla[u](x)$  will denote a so-called *minimal weak upper gradient* of the function  $u(\cdot)$ , i.e. the (measurable) minimal function such that for any rectifiable curve  $\gamma : [0, 1] \rightarrow X$

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g(s) ds.$$

Note that if the function  $u(\cdot)$  satisfies the Lipschitz condition:

$$|u(x) - u(y)| \leq L \cdot d(x, y), \quad 0 \leq L = \text{const} < \infty,$$

then the function  $g(x) = \nabla[u](x)$  there exists and is bounded:  $g(x) \leq L$ .

"The term Poincaré type inequality is used, somewhat loosely, to describe a class of inequalities that generalize the classical Poincaré inequality"

$$\int_D |u(z)|^p dx \leq A_m(p, D) \int_D |\text{grad } u(z)|^p dz, \quad A_m(p, D) = \text{const} < \infty, \quad (1.0)$$

see [1], chapter 8, p.215, and the source work of Poincaré [36].

A particular case done by Wirtinger: "an inequality ascribed to Wirtinger", see [25], p. 66-68; see also [33], [38], [39].

In the inequality (1.0)  $D$  may be for instance open bounded non empty convex subset of the whole space  $R^m$  and has a Lipschitz or at last Hölder boundary, or consists on the finite union of these domains, and  $|\text{grad } u(z)|$  is ordinary Euclidean  $R^m$  norm of "natural" distributive gradient of the differentiable a.e. function  $u$ .

The *generalized* Sobolev's norm, more exactly, semi-norm  $\|f\|_{W_p^1}$  of a "weak differentiable" function  $f : X \rightarrow R$  may be defined by the formula

$$\|f\|_{W_p^1} \stackrel{\text{def}}{=} \left[ \int_X |\nabla f|^p d\mu(x) \right]^{1/p} = \|\nabla f\|_p.$$

We will call "the Poincaré inequality", or more precisely "the Poincaré  $(L(p), L(q))$  inequality" more general inequalities of the forms

$$\begin{aligned} \mu(X)^{-1/q} \|u - u_X\|_q &\leq K_P(p, q) \text{diam}(X) \mu(X)^{-1/p} \|\nabla u\|_p = \\ &K_P(p, q) \text{diam}(X) \mu(X)^{-1/p} \|u\|_{W_p^1}, \end{aligned} \quad (1.1)$$

where

$$1 \leq p < s = \text{const} > 1, \quad 1 \leq q < \frac{ps}{s-p} - \quad (1.1a)$$

the Poincaré-Lebesgue-Riesz version; or in the case when  $p > s$  and after (possible) redefinition of the function  $u = u(x)$  on a set of measure zero

$$\begin{aligned} |u(x) - u(y)| &\leq K_L(s, p) d^{1-s/p}(x, y) \mu(X) \|\nabla u\|_p = \\ &K_L(s, p) \mu(X) \|u\|_{W_p^1} - \end{aligned} \quad (1.2)$$

the Poincaré-Lipshitz version; the case  $p = s$  in our setting of problem, indeed, in the terms of Orlicz's spaces and norms, is considered in [19].

The last inequality (1.2) may be reformulated in the terms of the module of continuity of the function  $u$  :

$$\omega(u, \tau) := \sup_{x, y: d(x, y) \leq \tau} |u(x) - u(y)|, \quad \tau \geq 0.$$

Namely,

$$\omega(u, \tau) \leq K_L(s, p) \tau^{1-s/p} \mu(X) \|u\|_{W_p^1}.$$

We will name following the authors of articles [13], [19] etc. all the spaces  $(X, M, \mu, d)$  which satisfied the inequalities (1.1) or (1.2) for each functions  $\{u\}$  having the weak gradient correspondingly as a *Poincaré-Lebesgue spaces* or *Poincaré-Lipshitz spaces*.

As for the constants  $s, K_P(s, p), K_L(s, p)$ . The value  $s$  may be defined from the following condition (if there exists)

$$\inf_{x_1, x_2 \in X} \left\{ \frac{\mu(B(x_1, r_1))}{\mu(B(x_2, r_2))} \right\} \geq C \cdot \left[ \frac{r_1}{r_2} \right]^s, \quad C = \text{const} > 0,$$

where as ordinary  $B(x, r)$  denotes a closed ball relative the distance  $d(\cdot, \cdot)$  with the center  $x$  and radii  $r, r > 0$ :

$$B(x, r) = \{y, y \in X, d(x, y) \leq r\}.$$

This condition is equivalent to the so-called double condition, see [13], [19] and is closely related with the notion of Ahlfors  $Q$  – regularity

$$C_1 r^Q \leq \mu(B(x, r)) \leq C_2 r^Q, \quad C_1, C_2, Q = \text{const} > 0,$$

see [19], [26].

Further, we will understand as a capacity of the values  $K_P(s, p), K_L(s, p)$  its minimal values, namely

$$K_P(p, q) \stackrel{\text{def}}{=} \sup_{0 < \|\nabla u\|_p < \infty} \left\{ \frac{\mu(X)^{-1/q} \|u - u_X\|_q}{\text{diam}(X) \mu(X)^{-1/p} \|\nabla u\|_p} \right\}, \quad (1.3a)$$

$$K_L(s, p) \stackrel{\text{def}}{=} \sup_{0 < \|\nabla u\|_p < \infty} \left\{ \frac{|u(x) - u(y)|}{d^{1-s/p}(x, y) \mu(X) \|\nabla u\|_p} \right\}. \quad (1.3b)$$

We will denote for simplicity

$$s = \text{order } X = \text{order}(X, M, \mu, d).$$

There are many publications about grounding of these inequalities under some conditions and about its applications, see, for instance, in articles [6], [7], [11], [13], [14], [19], [20], [23], [30], [34], [37], [40] and in the classical monographs [3], [12]; see also reference therein.

**Our aim is a generalization of the estimation (1.1) and (1.2) on the so-called Bilateral Grand Lebesgue Spaces  $BGL = BGL(\psi) = G(\psi)$ , i.e. when  $u(\cdot) \in G(\psi)$  and to show the precision of obtained estimations.**

We recall briefly the definition and needed properties of these spaces. More details see in the works [9], [10], [15], [16], [28], [29], [21], [17], [18] etc. More about rearrangement invariant spaces see in the monographs [4], [22].

For  $b = \text{const}, 1 < b \leq \infty$ , let  $\psi = \psi(p), p \in [1, b)$ , be a continuous positive function such that there exists a limits (finite or not)  $\psi(1 + 0)$  and  $\psi(b - 0)$ , with conditions  $\inf_{p \in (1, b)} \psi(p) > 0$  and  $\min\{\psi(1 + 0), \psi(b - 0)\} > 0$ . We will denote the set of all these functions as  $\Psi(b)$  and  $b = \text{supp } \psi$ .

The Bilateral Grand Lebesgue Space (in notation BGLS)  $G(\psi; a, b) = G(\psi)$  is the space of all measurable functions  $f : R^d \rightarrow R$  endowed with the norm

$$\|f\|_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (a, b)} \left[ \frac{|f|_p}{\psi(p)} \right], \quad (1.4)$$

if it is finite.

In the article [29] there are many examples of these spaces.

The  $G(\psi)$  spaces over some measurable space  $(X, M, \mu)$  with condition  $\mu(X) = 1$  (probabilistic case) appeared in [21].

The BGLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces  $L_1(R^d)$  and  $L_\infty(R^d)$  under real interpolation method [2], [5], [17], [18].

It was proved also that in this case each  $G(\psi)$  space coincides only under some additional conditions: convexity of the functions  $p \rightarrow p \cdot \ln \psi(p)$ ,  $b = \infty$  etc. [29] with the so-called *exponential Orlicz space*, up to norm equivalence.

In others quoted publications were investigated, for instance, their associate spaces, fundamental functions  $\phi(G(\psi; a, b); \delta)$ , Fourier and singular operators, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces, etc.

**Remark 1.1** If we introduce the *discontinuous* function

$$\psi_r(p) = 1, \quad p = r; \quad \psi_r(p) = \infty, \quad p \neq r, \quad p, r \in (a, b) \quad (1.5)$$

and define formally  $C/\infty = 0$ ,  $C = \text{const} \in R^1$ , then the norm in the space  $G(\psi_r)$  coincides formally with the  $L_r$  norm:

$$\|f\|_{G(\psi_r)} = |f|_r. \quad (1.5a)$$

Thus, the Bilateral Grand Lebesgue spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces  $L_r$ .

**Remark 1.2.** Let  $F = \{f_\alpha(x)\}$ ,  $x \in X$ ,  $\alpha \in A$  be certain family of numerical functions  $f_\alpha(\cdot) : x \rightarrow R$ ,  $A$  is arbitrary set, such that

$$\exists b > 1, \quad \forall p < b \Rightarrow \psi_F(p) \stackrel{\text{def}}{=} \sup_{\alpha \in A} \|f_\alpha(\cdot)\|_p < \infty. \quad (1.6)$$

The function  $p \rightarrow \psi_F(p)$  is named ordinary as *natural* function for the family  $F$ . Evidently,

$$\forall \alpha \in A \Rightarrow f_\alpha(\cdot) \in G\psi_F$$

and moreover

$$\sup_{\alpha \in A} \|f_\alpha(\cdot)\|_{G\psi_F} = 1. \quad (1.7)$$

The BGLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in PDE [9], [15], theory of probability in Banach spaces [24], [21], [28], in the modern non-parametrical statistics, for example, in the so-called regression problem [28].

We will denote as ordinary the indicator function

$$I(A) = I(x \in A) = 1, \quad x \in A, \quad I(x \in A) = 0, \quad x \notin A;$$

here  $A$  is a measurable set.

Recall, see, e.g. [4] that the fundamental function  $\phi(\delta, S)$ ,  $\delta > 0$  of arbitrary rearrangement invariant space  $S$  over  $(X, M, \mu)$  with norm  $\|\cdot\|_S$  is

$$\phi(\delta, S) \stackrel{\text{def}}{=} \|I(A)\|_S, \quad \mu(A) = \delta.$$

We have in the case of BGLS spaces

$$\phi(\delta, G\psi) = \sup_{1 \leq p < b} \left[ \frac{\delta^{1/p}}{\psi(p)} \right]. \quad (1.8)$$

This notion play a very important role in the functional analysis, operator theory, theory of interpolation and extrapolation, theory of Fourier series etc., see again [4]. Many properties of the fundamental function for BGLS spaces with considering of several examples see in the articles [29], [27].

**Example 1.1.** Let  $\mu(X) = 1$  and let

$$\psi^{(b,\beta)}(p) = (b-p)^{-\beta}, \quad 1 \leq p < b, \quad b = \text{const} > 1, \quad \beta = \text{const} > 0, \quad (1.9)$$

then as  $\delta \rightarrow 0+$

$$\phi(G\psi^{(b,\beta)}, \delta) \sim (\beta b^2/e)^\beta \cdot \delta^{1/b} \cdot |\ln \delta|^{-\beta}. \quad (1.9a)$$

**Example 1.2.** Let again  $\mu(X) = 1$  and let now

$$\psi_{(\beta)}(p) = p^\beta, \quad 1 \leq p < \infty, \quad \beta = \text{const} > 0, \quad (1.10)$$

then as  $\delta \rightarrow 0+$

$$\phi(G\psi_{(\beta)}, \delta) \sim \beta^\beta |\ln \delta|^{-\beta}. \quad (1.10a)$$

## 2. MAIN RESULT: BGLS ESTIMATIONS FOR POINCARÉ-LEBESGUE-RIESZ VERSION. THE CASE OF PROBABILITY MEASURE.

We suppose in this section without loss of generality that the measure  $\mu$  is probabilistic:  $\mu(X) = 1$  and that the source tetrad  $(X, M, \mu, d)$  is Poincaré-Lebesgue space.

Assume also that the function  $|\nabla u(x)|$ ,  $x \in X$  belongs to certain BGLS  $G\psi$  with  $\text{supp } \psi = s = \text{order } X > 1$ ; the case when  $\text{order } \psi = b \neq s$  may be reduced to considered here by transfiguration  $s' := \min(b, s)$ .

The function  $\psi(\cdot)$  may be constructively introduced as a natural function for one function  $|\nabla u|$ :

$$\psi_{(0)}(p) := \|u\|_{W_p^1},$$

if there exists and is finite for at least one value  $p$  greatest than one.

Define the following function from the set  $\Psi$

$$\nu(q) := \inf_{p \in (qs/(q+s), s)} \{K_P(p, q) \cdot \psi(p)\}, \quad 1 \leq q < \infty. \quad (2.1)$$

**Proposition 2.1.**

$$\|u - u_X\|_{G\nu} \leq \text{diam}(X) \cdot \|\nabla u\|_{G\psi}, \quad (2.2)$$

where the "constant"  $\text{diam}(X)$  is the best possible.

**Proof.** We can suppose without loss of generality  $\|\nabla u\|_{G\psi} = 1$ , then it follows by the direct definition of the norm in BGLS

$$\|\nabla u\|_p \leq \psi(p), \quad 1 \leq p < s. \quad (2.3)$$

The inequality (1.1) may be rewritten in our case as follows:

$$\|u - u_X\|_q \leq K_P(p, q) \cdot \text{diam}(X) \cdot \|\nabla u\|_p,$$

therefore

$$\|u - u_X\|_q \leq K_P(p, q) \cdot \text{diam}(X) \cdot \psi(p), \quad 1 \leq p < s. \quad (2.4)$$

Since the value  $p$  is arbitrary in the set  $1 \leq p < s$ , we can take the minimum of the right - hand side of the inequality (2.4):

$$\|u - u_X\|_q \leq \text{diam}(X) \cdot \inf_{1 \leq p < s} [K_P(p, q) \cdot \psi(p)] = \text{diam}(X) \cdot \nu(q),$$

which is equivalent to the required estimate

$$\|u - u_X\|_{G\nu} \leq \text{diam}(X) = \text{diam}(X) \|\nabla u\|_{G\psi}.$$

The exactness of the constant  $\text{diam}(X)$  in the inequality (2.2) follows immediately from theorem 2.1 in the article [32].

### 3. MAIN RESULT: BGLS ESTIMATIONS FOR POINCARÉ-LEBESGUE-RIESZ VERSION. THE GENERAL CASE OF ARBITRARY MEASURE.

The case when the value  $\mu(X)$  is variable, is more complicated. As a rule, in the role of a sets  $X$  acts balls  $B(x, r)$ , see [13], [19].

**Definition 3.1.** We will say that the function  $K_P(p, q)$ ,  $1 \leq p < s, 1 \leq q < \infty$  allows factorable estimation, symbolically:  $K_P(\cdot, \cdot) \in AFE$ , iff there exist two functions  $R = R(p) \in \Psi(s)$  and  $V = V(q) \in G\Psi(\infty)$  such that

$$K_P(p, q) \leq R(p) \cdot V(q). \quad (3.1)$$

**Theorem 3.1.** Suppose that the source tetrad  $(X, M, \mu, d)$  is again Poincaré-Lebesgue space such that  $K_P(\cdot, \cdot) \in AFE$ . Let  $\zeta = \zeta(q)$  be arbitrary function from the set  $\Psi(\infty)$ .

Assume also as before in the second section that the function  $|\nabla u(x)|$ ,  $x \in X$  belongs to certain BGLS  $G\psi$  with  $\text{supp } \psi = s = \text{order } X > 1$ ; the case when  $\text{order } \psi = b \neq s$  may be reduced to considered here by transfiguration  $s' := \min(b, s)$ .

Our statement:

$$\frac{\|u - u_X\|_{G(V \cdot \zeta)}}{\phi(G\zeta, \mu(X))} \leq \text{diam}(X) \cdot \frac{\|\nabla u\|_{G\psi}}{\phi(R \cdot \psi, \mu(X))} \quad (3.2)$$

and the "constant"  $\text{diam}(X)$  in (3.2) is as before the best possible.

**Proof.** Denote and suppose for brevity  $u^{(0)} = u - u_X$ ,  $\mu = \mu(X)$ ,  $\text{diam}(X) = 1$ ,  $\|\nabla u\|_{G\psi} = 1$ . The last equality imply in particular

$$\|\nabla u\|_p \leq \psi(p), \quad 1 \leq p < s. \quad (3.3)$$

The inequality (1.1) may be reduced taking into account (3.3) as follows

$$\mu^{-1/q} \|u^{(0)}\|_q \leq R(p) V(q) \mu(X)^{-1/p} \psi(p),$$

and after dividing by  $\zeta(q)$  and by  $\mu^{-1/q}$

$$\frac{\|u^{(0)}\|_q}{V(q)\zeta(q)} \leq R(p) \psi(p) \mu^{-1/p} \frac{\mu^{1/q}}{\zeta(q)}. \quad (3.4)$$

We take the supremum from both the sides of (3.4) over  $q$  using the direct definition of the fundamental function and norm for BGLS:

$$\|u^{(0)}\|_{G(V \cdot \zeta)} \leq \frac{R(p) \psi(p)}{\mu^{1/p}} \cdot \phi(G\zeta, \mu). \quad (3.5)$$

Since the left-hand side of relation (3.5) does not depend on the variable  $p$ , we can take the infimum over  $p$ . As long as

$$\inf_p \left[ \frac{R(p) \psi(p)}{\mu^{1/p}} \right] = \left[ \sup_p \frac{\mu^{1/p}}{R(p) \psi(p)} \right]^{-1} = [\phi(G(R \cdot \psi), \mu)]^{-1},$$

we deduce from (3.5)

$$\frac{\|u^{(0)}\|_{G(V \cdot \zeta)}}{\phi(G\zeta, \mu)} \leq \frac{1}{\phi(G(R \cdot \psi), \mu)} = \text{diam } X \cdot \frac{\|\nabla u\|_{G\psi}}{\phi(G(R \cdot \psi), \mu)},$$

Q.E.D.

#### 4. MAIN RESULT: BGLS ESTIMATIONS FOR POINCARÉ-LIPSCHITZ VERSION.

Recall that we take the number  $s$ ,  $s > 1$  to be constant.

We consider in this section the case when  $p \in (s, b)$ ,  $s < b = \text{const} \leq \infty$ .

**Theorem 4.1.** Suppose the fourth  $(X, M, \mu, d)$  is Poincaré-Lipshitz space and that the function  $|\nabla u(x)|$ ,  $x \in X$  belongs to certain BGLS  $G\psi$  with  $\text{supp } \psi = b$ . Then the function  $u = u(x)$  satisfies after (possible) redefinition on a set of measure zero the inequality

$$|u(x) - u(y)| \leq \mu(X) \cdot \frac{d(x, y)}{\phi(G(K_L \cdot \psi), d^s(x, y))} \cdot \|\nabla u\|_{G\psi}, \quad (4.1)$$

or equally

$$\omega(u, \tau) \leq \mu(X) \cdot \frac{\tau}{\phi(G(K_L \cdot \psi), \tau^s)} \cdot \|\nabla u\|_{G\psi}, \quad (4.1a)$$

and this time the "constant"  $\mu(X)$  is best possible.

**Proof.** Suppose for brevity  $\mu(X) = 1$ ,  $\|\nabla u\|G\psi = 1$ . The last equality imply in particular

$$\|\nabla u\|_p \leq \psi(p), \quad s < p < b. \quad (4.2)$$

The function  $u(\cdot)$  satisfies the inequality (1.2) after (possible) redefinition of the function  $u = u(x)$  on a set of measure zero

$$|u(x) - u(y)| \leq K_L(s, p) d^{1-s/p}(x, y) \mu(X) \|\nabla u\|_p \leq K_L(s, p) \psi(p) \cdot d^{1-s/p}(x, y). \quad (4.3)$$

The excluding set in (4.3) may be dependent on the value  $p$ , but it sufficient to consider this inequality only for the rational values  $p$  from the interval  $(s, b)$ .

The last inequality may be transformed as follows

$$\frac{|u(x) - u(y)|}{d(x, y)} \leq \frac{K_L(s, p) \cdot \psi(p)}{d^{s/p}(x, y)} = \left[ \frac{d^{s/p}(x, y)}{K_L(s, p) \cdot \psi(p)} \right]^{-1}. \quad (4.4)$$

Since the left - hand side of (4.4) does not dependent on the variable  $p$ , we can take the infimum from both all the sides of (4.4):

$$\frac{|u(x) - u(y)|}{d(x, y)} \leq \left[ \sup_p \left\{ \frac{d^{s/p}(x, y)}{K_L(s, p) \cdot \psi(p)} \right\} \right]^{-1} = \frac{1}{\phi(G(K_L \cdot \psi), d^s(x, y))} = \frac{\mu(X)}{\phi(G(K_L \cdot \psi), d^s(x, y))},$$

or equally

$$|u(x) - u(y)| \leq \mu(X) \cdot \frac{d(x, y)}{\phi(G(K_L \cdot \psi), d^s(x, y))} = \mu(X) \cdot \frac{d(x, y)}{\phi(G(K_L \cdot \psi), d^s(x, y))} \cdot \|\nabla u\|G\psi. \quad (4.5)$$

The exactness of the constant  $\mu(X)$  may be proved as before, by mention of the article [32].

This completes the proof of theorem 4.1.

Let us consider two examples.

**Example 4.1.** Suppose in addition to the conditions of theorem 4.1 that  $\mu(X) = 1$  and

$$K_L(s, p) \cdot \psi(p) = \psi^{(b, \beta)}(p) = (b - p)^{-\beta}, \quad 1 \leq p < b, \quad b = \text{const} > 1, \quad \beta = \text{const} > 0.$$

We deduce taking into account the example 1.1 that for almost everywhere values  $(x, y)$  and such that  $d(x, y) \leq 1/e$

$$|u(x) - u(y)| \leq C_1(b, \beta, s) d^{1-1/b}(x, y) |\ln d(x, y)|^\beta \cdot \|\nabla u\|G\psi.$$

**Example 4.2.** Suppose in addition to the conditions of theorem 4.1 that  $\mu(X) = 1$  and



$$K_L(s, p) \cdot \psi(p) = \psi_{(\beta)}(p) = p^\beta, \quad 1 \leq p < \infty, \quad \beta = \text{const} > 0.$$

We deduce taking into account the example 1.2 that for almost everywhere values  $(x, y)$  and such that  $d(x, y) \leq 1/e$

$$|u(x) - u(y)| \leq C_2(\beta, s) d(x, y) |\ln d(x, y)|^\beta \|\nabla u\| G\psi.$$

## 5. CONCLUDING REMARKS

**A.** It may be interest by our opinion to investigate the *weights* generalization of obtained inequalities, alike as done for the classical Sobolev's case, see for instance [6], [23], [35], [37].

**B.** The physical applications of these inequalities, for example, in the uncertainty principle. is described in the article of C.Fefferman [8].

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